11/8/25

53 mins

4.5/6

= 75%

1. (a) Give a general expression for the quadratic approximation to a twice differentiable function $f(x)$ at $x = a$.

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

(b) Use your answer from part (a) to give an approximate value for $\ln(1.2)$, where $\ln(x)$ is the natural log function.

$$\ln(1.2) \approx \ln 1 + f'(1)(1.2-1) + \frac{f''(1)}{2}(1.2-1)^2$$

$$= 0 + \frac{1}{1}0.2 + \frac{\left(-\frac{1}{1}\right)}{2}0.2^2$$

$$= 0.2 - 0.02$$

$$= 0.18$$

2. Salt is poured from a conveyer belt at a rate of 30 ft^3/min, forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 ft. high?

$V = \frac{1}{3}\pi r^2 h$, $\frac{dV}{dt} = 30 \text{ft}^3/\text{min}$, $h = 2r$

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

$= \frac{h^3}{6}\pi$ $\quad \frac{\pi}{12}h^3$

$\frac{dV}{dh} = \frac{\pi}{2}h^2$

0.5

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$30 = \frac{\pi}{2}(10)^2 \times \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{3}{5\pi}$

X

3. Draw a careful picture of the graph of the function

$$f(x) = x - 3x^{1/3}.$$

Be sure to indicate the coordinates of any local maxima and minima, the intervals on which the function is increasing and decreasing, and asymptotes (if any of these features occur). Computing inflection points may help you draw an accurate picture, but is not necessary.

$f(x) = x - 3x^{\frac{1}{3}}$

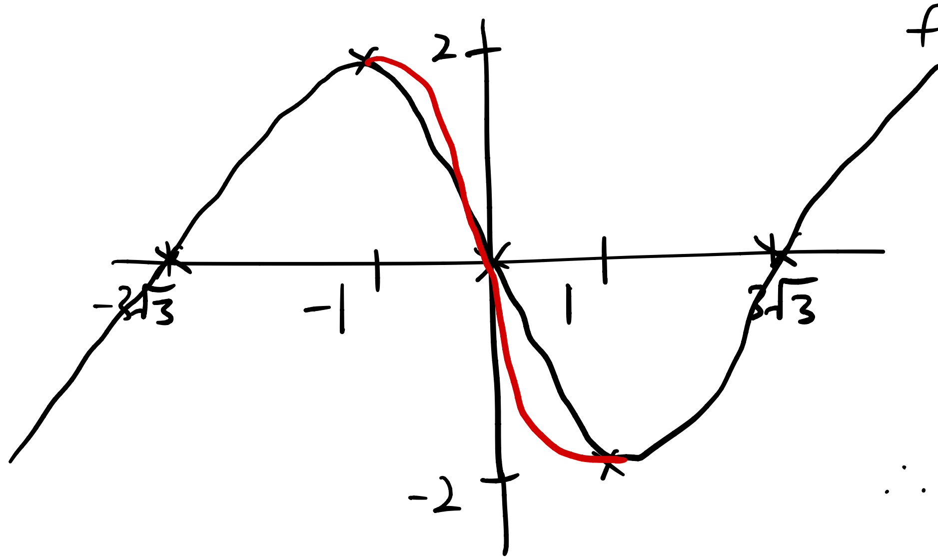
$= x^{\frac{1}{3}}(x^{\frac{2}{3}} - 3)$

$x^{\frac{1}{3}} = 0 \Rightarrow x = 0$

$x^{\frac{2}{3}} = 3 \Rightarrow x^2 = 3^3$

$x = \pm 3\sqrt{3}$

$\therefore f(x) = 0$ at $x = 0, \pm 3\sqrt{3}$



$f'(x) = 1 - x^{-\frac{2}{3}} = 1 - \frac{1}{x^{\frac{2}{3}}}$

$f'(x) = 0 \Rightarrow 1 - \frac{1}{x^{\frac{2}{3}}} = 0$

$x^{\frac{2}{3}} = 1$

$x^2 = 1^3$

$x = \pm 1$

$f(1) = 1 - 3 = -2$

$f(-1) = -1 - 3(-1) = 2$

As $x \to \infty$, $f(x) \to \infty$.

As $x \to -\infty$, $f(x) \to -\infty$

0.5

Tangent line to the graph is vertical at $x = 0$

$f'(x) = 1 - \frac{1}{x^{\frac{2}{3}}}$

$f'(x) > 0 \Rightarrow \frac{1}{x^{\frac{2}{3}}} < 1$

$x^{\frac{2}{3}} > 1$

$x^2 > 1$

$x > 1, x < -1$

$\therefore f$ is increasing at $(-\infty, -1)$ and $(1, \infty)$

$f'(x) < 0 \Rightarrow \frac{1}{x^{\frac{2}{3}}} > 1$

$\Rightarrow x^2 < 1$

$x > -1, x < 1$

$\therefore f$ is decreasing at $-1 < x < 1$.

4. A metal storage tank is to be made in the shape of a cylinder with a circular base and a hemispherical top. Find the dimensions of the tank which require the least amount of metal used to hold a fixed constant volume V.

$$V = \frac{2}{3}\pi r^3 + \pi r^2 h \qquad \Rightarrow \pi r^2 h = V - \frac{2}{3}\pi r^3$$

$$h = \frac{V}{\pi r^2} - \frac{2}{3} r$$

$$SA = 2\pi r^2 + 2\pi r h + \pi r^2$$

$$= 3\pi r^2 + 2\pi r h$$

$$\frac{dh}{dr} = -\frac{2V}{\pi r^3} - \frac{2}{3}$$

$$\frac{dSA}{dr} = 6\pi r + 2\pi \left(h + r\frac{dh}{dr}\right)$$

$$= 6\pi r + 2\pi \left(\frac{V}{\pi r^2} - \frac{2}{3} r - \frac{2V}{\pi r^2} - \frac{2}{3} r\right)$$

$$= 6\pi r + 2\pi \left(-\frac{V}{\pi r^2} - \frac{4}{3} r\right)$$

$$= 6\pi r - \frac{2V}{r^2} - \frac{8\pi r}{3}$$

$$= \frac{10\pi r}{3} - \frac{2V}{r^2}$$

$$\frac{dSA}{dr} = 0 \Rightarrow \frac{10\pi r}{3} - \frac{2V}{r^2} = 0$$

$$\frac{10\pi}{3} r^3 = 2V$$

$$r^3 = \frac{3V}{5\pi}$$

$$\pi r^2 = \frac{3V}{5r}$$

$$\Rightarrow h = \frac{V}{\frac{3V}{5r}} - \frac{2}{3} r$$

$$= \frac{5r}{3} - \frac{2}{3} r$$

$$= r$$

$$\therefore r = h = \sqrt[3]{\frac{3V}{5\pi}}$$

$SA = 3\pi r^2 + 2\pi r h$

$= 3\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2} - \frac{2}{3}r\right)$

As $r \to 0$, $SA \to \infty$.

As $r \to \infty$, $SA \to \infty$.

$\because$ SA has only one critical point at $r = h = \sqrt[3]{\frac{3V}{5\pi}}$, then it must be a minimum.

$\therefore$ The least amount of material required is $r = h = \sqrt[3]{\frac{3V}{5\pi}}$.

5. Explain why Newton's method eventually fails when finding zeroes of $f(x) = x^3 - 3x + 7$ with a starting value $x_1 = 2$.

$$f(x) = x^3 - 3x + 7$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_2 = 2 - \frac{9}{9} = 1$$

$$x_3 = 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \frac{5}{0}$$

$$f(2) = 8 - 6 + 7 = 9$$

$$f'(x) = 3x^2 - 3$$

$$f'(2) = 12 - 3 = 9$$

$$f(1) = 1 - 3 + 7 = 5$$

$$f'(1) = 0$$

$\therefore$ It fails because on iteration 2, $f'(x) = 0$.

6. Prove that

$$\sqrt{1+x} < 1 + \frac{1}{2}x, \quad \text{if } x > 0.$$

Let $f(x) = \sqrt{1+x} - \left(1 + \frac{1}{2}x\right)$.

$f(0) = 1 - 1 = 0$

$f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+x}} - \frac{1}{2} \Rightarrow f'(x) < 0$ for $x > 0$. elaborate more

$\Rightarrow f$ is decreasing for $x > 0$.

$\Rightarrow f(x) < f(0)$ for $x > 0$

$\sqrt{1+x} - \left(1 + \frac{1}{2}x\right) < 0 \Leftrightarrow \sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$. $\square$

0.5